# Cliffonic dressing of Lie algebras and superalgebras

Valeriy N. Tolstoy

Lomonosov Moscow State University
Skobelsyn Institute of Nuclear Physics
119 992 Moscow, Russia

November 13, 2025

Abstract

It is shown that basic classical Lie superalgebras can be extended to $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$-graded superalgebras with the same even part of the $\mathbb{Z}_2$-case. For this purpose, a $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$-graded bosonic algebra is introduced using a cliffonic dressing of an ordinary bosonic algebra. Some real forms of such $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$-superalgebras are considered. A new concept of Hermitian imaginary unit is introduced (the usual imaginary unit is an anti-Hermitian one).

*Contents:*

*1. Introduction*

*2. Clifford algebra and cliffonic dressing*

*(2a) Cliffonic dressing for the construction of real forms*

*(2b) Cliffonic cross-dressing to change Lie (super)brackets*

*3. The complex* $\mathbb{Z}_2$*- and* $\mathbb{Z}_2\times\mathbb{Z}_2$*-graded Lie superalgebras*

*(3a) The complex* $\mathbb{Z}_2$*-graded Lie superalgebra*

*(3b) The complex* $\mathbb{Z}_2\times\mathbb{Z}_2$*-graded Lie superalgebra*

*4. Relationship of* $\mathfrak{osp}(1|4;\mathbb{C})$ *with* $\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C})$ *via cliffonic dressing*

*5. Relationship of* $\mathfrak{osp}(1|4;\mathbb{C})$ *with* $\tilde{\mathfrak{osp}}(1|2,2;\mathbb{C})$ *via cliffonic dressing*

*6. Real forms AdS*$_5$ *and dS*$_5$ *for* $\mathbb{Z}_2$*- and* $\mathbb{Z}_2\times\mathbb{Z}_2$ *- supersymmetries*

*Application A.* $\mathbb{Z}_2$*- and* $\mathbb{Z}_2\times\mathbb{Z}_2$*-graded Poincarè superalgebras*

*Application B.* $\mathbb{Z}_2$*- and* $\mathbb{Z}_2\times\mathbb{Z}_2$*-graded superspaces and superfields*

*Application C. Once more about SUGRA*

The orthosymplectic superalgebra $\mathfrak{osp}(1|4;\mathbb{C})$ can be represented as a direct sum of spaces

$$\mathfrak{osp}(1|4;\mathbb{C}) = \mathfrak{osp}_1(1|2;\mathbb{C}) \oplus \mathcal{P}_{12} \oplus \mathfrak{osp}_2(1|2;\mathbb{C}), \tag{1}$$

where the linear envelope

$$\mathcal{P}_{12} = \mathrm{Lin}\{X_{12}, X_{-1-2}, X_{1-2}, X_{-12}\} \tag{2}$$

is called the complex space of a curved four-momentum.
The orthosymplectic $\mathbb{Z}_2$-graded supersubalgebras $\mathfrak{osp}_i(1|2;\mathbb{C})$ $(i = 1, 2)$ are generated by the odd elements $X_{\pm i}$. In the case of $\mathbb{Z}_2$-grading, the elements of $\mathcal{P}_{12}$ are defined through the anticommutator $\{X_{\pm 1}.X_{\pm 2}\}$. If we consider the case $\mathbb{Z}_2 \times \mathbb{Z}_2$-graduation then the elements of $\mathcal{P}_{12}$ are defined through the commutator $[X_{\pm 1}.X_{\pm 2}]$.
We can easily generalize these results to the general $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$-graded superalgebras $\mathfrak{osp}(1|2n;\mathbb{C})$.

By definition the Clifford algebra $Cl_n(p,q)$ of the rank $n=p+q$, with the signature $(p,q)$, is a real or complex unital (with a unite) associative algebra generated by the generators $c_i$, $(i=1,2,\ldots,n)$, which will be called the *cliffons*, with the defining relations

$$\{c_i,c_j\}:=c_ic_j+c_jc_i=\eta_{ij}e,\quad i,j=1,\ldots n, \tag{3}$$

where $e$ is identity element, $\eta=||\eta_{ij}||=\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ is the diagonal $(n\times n)$-matrix with its first $p$ entries equal to 1 and the last $q$ entries equal to -1 on the diagonal. Sometimes we will use an involutive antiautomorphysm (or automorphism) $(^{\ast})$ which acts on the generating cliffons as follows: $(c_i^{\ast})^{\ast}=c_i$, $(c_ic_j\cdots c_k)^{\ast}=c_k^{\ast}\cdots c_j^{\ast}c_i^{\ast}$ (or $(c_ic_j\cdots c_k)^{\ast}=c_i^{\ast}c_j^{\ast}\cdots c_k^{\ast}$) for $i,j,\ldots,k=1,2,\ldots,n$.
The Clifford (cliffonic[1]) algebras $Cl_n(p,q)$ with the different stars are different and they have the different applications. We consider here a special $Cl_n(p,q)$-application called *cliffonic dressing*. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two different non-isomorphic (real or complex) Lie (super)algebras (or associative (super)algebras) of the same dimension. Cliffonic dressing is an exact (non-degenerate) special homomorphism $\mathfrak{g}\mapsto Cl_n(p,q)\otimes\mathfrak{g}'$ for a some clifonic algebra $Cl_n(p,q)$. In other words, we get the realization of algebra $\mathfrak{g}$ by "dressing" another equidimensional algebra $\mathfrak{g}'$ with the help of $Cl_n(p,q)$ cliffons.

[1]Below we will also use the name "cliffonic algebra" by analogy with bosonic and fermionic associative algebras.

# (2a) *Cliffonic dressing for the construction of real forms*

Let's consider a cliffonic algebra $Cl_{\bar{n}} := Cl_n(0, n)$ with a negative signature $\eta = ||\eta_{ij}|| = \mathrm{diag}(-1, -1, \ldots, -1)$, and with the following real cliffons: $(c_i c_j \cdots c_k)^* = c_i^* c_j^* \cdots c_k^* = c_i c_j \cdots c_k$ for $i, j, \ldots, k = 1, 2, \ldots, n$.
In the case of a one-cliffonic algebra $Cl_{\bar{1}}$, we put $J := c_1$ and we have $J = -1$ and $J^* = J$. The element $J$ with such properties is named the operator of a fundamental symmery in the theory of Krein spaces.
It is easy to see that the cliffonic algebras $Cl_n(n, 0)$ and $Cl'_n(p, q)$ with different signatures and the same star are connected to each other by means of $Cl_{\bar{1}}$-dressing, namely,

$$\begin{aligned} c'_i &= ec_i, \quad i = 1, \ldots, p, \\ c'_i &= Jc_i, \quad i = p+1, \ldots, n, \end{aligned} \tag{4}$$

or the inverse $Cl_{\bar{1}}$-dressing

$$\begin{aligned} c_i &= ec'_i, \quad i = 1, \ldots, p, \\ c_i &= Jc'_i, \quad i = p+1, \ldots, n, \end{aligned} \tag{5}$$

where $c_i$ $(i = 1, \ldots, n)$ are the cliffons of $Cl_n(n, 0)$ and $c'_i$ $(i = 1, \ldots, n)$ are the cliffons of $Cl'_n(p, q)$ and everywhere we use short notation $ec_i := e \otimes c_i$, $Jc_i := J \otimes c_i$.
In the following sections we will use the cliffonic dressing to construct real de Sitter and anti-de Sitter superalgebras.

In this special case we will use the cliffonic algebra $Cl_n := Cl_n(n,0)$ with a positive signature $\eta = ||\eta_{ij}|| = \mathrm{diag}(1,1,\ldots,1)$. We will apply the $Cl_n$-dressing to the well-known bosonic and fermionic associative algebras. Recall that the bosonic algebra $B_\Lambda$ ($\Lambda \in \mathbb{N}$) generated by the creation $b_i^+$ and annihilation $b_i$ ($i = 1,2,\ldots,\Lambda$) operators with the following defining relations:

$$\begin{aligned} &[b_i, b_j] = [b_i^+, b_j^+] = 0, \\ &[b_i, b_j^+] = \delta_{ij} I \end{aligned} \tag{6}$$

for $i,j = 1,2,\ldots,\Lambda$. Analogously, the fermionic algebra $F_\Lambda$ generated by the creation $a_i^+$ and annihilation $a_i$ ($i = 1,2,\ldots,\Lambda$) operators with the following defining relations:

$$\begin{aligned} &\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0, \\ &\{a_i, a_j^+\} = \delta_{ij} I \end{aligned} \tag{7}$$

for $i,j = 1,2,\ldots,\Lambda$.

Now we applay the $Cl_n$-cliffonic dressing to the bosonic and fermion algebras. Let $[\lambda_1, \lambda_2, \ldots, \lambda_n]$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1$, be the partition of a closed integer interval $[1, \Lambda]$ where $\Lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. We set for the the bosonic algebra $B_\Lambda$:

$$\begin{array}{lll}
\tilde{b}_{\lambda_1 j} = c_1 b_j, & \tilde{b}^+_{\lambda_1 j} = c_1 b^+_j, & j = 1, 2, \ldots, \lambda_1, \\
\tilde{b}_{\lambda_2 j} = c_2 b_j, & \tilde{b}^+_{\lambda_2 j} = c_2 b^+_j, & j = \lambda_1 + 1, \ldots, \lambda_1 + \lambda_2, \\
\ldots \quad \ldots & \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \\
\tilde{b}_{\lambda_n j} = c_n b_j, & \tilde{b}^+_{\lambda_n j} = c_n b^+_j, & j = \Lambda_{n-1} + 1, \ldots, \Lambda_{n-1} + \lambda_n,
\end{array} \tag{8}$$

and also $\tilde{I} = eI$, where $\Lambda_{n-1} := \sum_{s=1}^{n-1} \lambda_s$. It is easy to check that the new generators (8) satisfy the following relations (instead of (31)) for the given partition $[\lambda_1, \lambda_2, \ldots, \lambda_n]$:

$$\{\tilde{b}_{\lambda_r i}, \tilde{b}_{\lambda_{r'} j}\} = \{\tilde{b}^+_{\lambda_r i}, \tilde{b}^+_{\lambda_{r'} j}\} = \{\tilde{b}_{\lambda_r i}, \tilde{b}^+_{\lambda_{r'} j}\} = 0 \tag{9}$$

for $r \neq r'$, and

$$\begin{aligned}
&[\tilde{b}_{\lambda_r i}, \tilde{b}_{\lambda_r j}] = [\tilde{b}^+_{\lambda_r i}, \tilde{b}^+_{\lambda_r j}] = 0, \\
&[\tilde{b}_{\lambda_r i}, \tilde{b}^+_{\lambda_r j}] = \delta_{ij} \tilde{I},
\end{aligned} \tag{10}$$

for $r = 1, 2, \ldots, n$ and $i, j = \Lambda_r + 1, \Lambda_r + 2, \ldots, \Lambda_r + \lambda_{r+1}$, where $\Lambda_r := \sum_{s=1}^{r} \lambda_s$.

In the case of a complete cliffonic cross-dressing, when $n = \Lambda$, $\Lambda_r = r$ $(r = 1, 2, \ldots, \Lambda)$, the formulas (8) take the form

$$\tilde{b}_j = c_j b_j, \quad \tilde{b}_j^+ = c_j b_j^+, \quad j = 1, 2, \ldots, \Lambda. \tag{11}$$

These cross-dressing generators satisfy the defining relations:

$$\{\tilde{b}_i, \tilde{b}_j\} = \{\tilde{b}_i^+, \tilde{b}_j^+\} = \{\tilde{b}_i, \tilde{b}_j^+\} = 0, \tag{12}$$

for $i \neq j$, $(i, j = 1, 2, \ldots, \Lambda)$, and

$$[\tilde{b}_i, \tilde{b}_i^+] = \tilde{I} \tag{13}$$

for $i = 1, 2, \ldots, \Lambda$.

In the case of a complete cliffonic cross-dressing, when $n = \Lambda$, $\Lambda_r = r$ $(r = 1, 2, \ldots, \Lambda)$, the formulas (8) take the form

$$\tilde{a}_j = c_j a_j, \quad \tilde{a}_j^+ = c_j a_j^+, \quad j = 1, 2, \ldots, \Lambda. \tag{14}$$

The cross-dressing generators (17) satisfy the defining relations:

$$[\tilde{a}_i, \tilde{a}_j] = [\tilde{a}_i^+, \tilde{a}_j^+] = [\tilde{a}_i, \tilde{a}_j^+] = 0 \tag{15}$$

for $i \neq j$, $(i, j = 1, 2, \ldots, \Lambda)$, and

$$\begin{aligned} \{\tilde{a}_i, \tilde{a}_i\} &= \{\tilde{a}_i^+, \tilde{a}_i^+\} = 0, \\ \{\tilde{a}_i, \tilde{a}_i^+\} &= \tilde{I} \end{aligned} \tag{16}$$

for $i = 1, 2, \ldots, \Lambda$.

Take once again the standard fermionic algebra $F_n$ generated by the creation and annihilation operators $a_i^+$, $a_i$ $(i = 1, 2, \ldots, n)$

$$\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0, \quad \{a_i, a_j^+\} = \delta_{ij} I, \quad i = 1, 2, \ldots, n. \qquad (17)$$

Let us introduce the new generators:

$$c_{2i-1} := (a_i + a_i^+), \quad c_{2i} := \imath(a_i - a_i^+), \quad i = 1, 2, \ldots, n. \qquad (18)$$

These new generators $c_i$ $(i = 1, 2, \ldots, 2n)$ satisfy the Clifford algebra defining relations

$$\{c_i, c_j\} := c_i c_j + c_j c_i = \eta_{ij} e, \quad i, j = 1, \ldots 2n, \qquad (19)$$

where $e$ is identity element, $\eta = ||\eta_{ij}|| = \mathrm{diag}(1, 1, \ldots, 1)$ is the diagonal $(2n \times 2n)$-matrix.

## (3a) The complex $\mathbb{Z}_2$-graded superalgebra

The complex $\mathbb{Z}_2$-graded Lie superalgebra (LSA) $\mathfrak{g}$, as a linear space over $\mathbb{C}$, is a direct sum of two graded components

$$\mathfrak{g} = \bigoplus_{a=0,1} \mathfrak{g}_a = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{1}$$

with a bilinear operation (the general Lie bracket), $[\![\cdot,\cdot]\!]$, satisfying the identities:

$$\deg([\![x_a, y_b]\!]) = \deg(x_a) + \deg(x_b) = a + b \ \ (\bmod 2), \tag{2}$$

$$[\![x_a, y_b]\!] = -(-1)^{ab}[\![y_b, x_a]\!], \tag{3}$$

$$[\![x_a, [\![y_b, z]\!]]\!] = [\![[\![x_a, y_b]\!], z]\!] + (-1)^{ab}[\![y_b, [\![x_a, z]\!]]\!], \tag{4}$$

where the elements $x_a$ and $y_b$ are homogeneous, $x_a \in \mathfrak{g}_a$, $x_b \in \mathfrak{g}_b$, and the element $z \in \mathfrak{g}$ is not necessarily homogeneous. The grading function $\deg(\cdot)$ is defined for homogeneous elements of the subspaces $\mathfrak{g}_0$ and $\mathfrak{g}_1$ modulo 2, $\deg(\mathfrak{g}_0) = 0$, $\deg(\mathfrak{g}_1) = 1$. The first identity (2) is called the grading condition, the second identity (3) is called the symmetry property and the condition (4) is the Jacobi identity. From (2) and (3) it is follows that the general Lie bracket $[\![\cdot,\cdot]\!]$ for homogeneous elements posses two value: commutator $[\cdot,\cdot]$ and anticommutator $\{\cdot,\cdot\}$. From (2) it is follows that $\mathfrak{g}_0$ is a Lie subalgebra in $\mathfrak{g}$, and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module:

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1, \quad (\{\mathfrak{g}_1, \mathfrak{g}_1\} \subseteq \mathfrak{g}_0). \tag{5}$$

The complex $\mathbb{Z}_2\times\mathbb{Z}_2$-graded LSA $\tilde{\mathfrak{g}}$, as a linear space, is a direct sum of four graded components

$$\tilde{\mathfrak{g}} = \bigoplus_{\mathrm{a}=(a_1,a_2)} \tilde{\mathfrak{g}}_{\mathrm{a}} = \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)} \oplus \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)} \tag{6}$$

with a bilinear operation $[\![\cdot,\cdot]\!]$ satisfying the identities (grading, symmetry, Jacobi):

$$\deg([\![x_{\mathrm{a}},y_{\mathrm{b}}]\!]) = \deg(x_{\mathrm{a}})+\deg(x_{\mathrm{b}}) = \mathrm{a}+\mathrm{b} = (a_1+b_1, a_2+b_2), \tag{7}$$

$$[\![x_{\mathrm{a}},y_{\mathrm{b}}]\!] = -(-1)^{\mathrm{ab}}[\![y_{\mathrm{b}},x_{\mathrm{a}}]\!], \tag{8}$$

$$[\![x_{\mathrm{a}},[\![y_{\mathrm{b}},z]\!]]\!] = [\![[\![x_{\mathrm{a}},y_{\mathrm{b}}]\!],z]\!] + (-1)^{\mathrm{ab}}[\![y_{\mathrm{b}},[\![x_{\mathrm{a}},z]\!]]\!], \tag{9}$$

where the vector $(a_1+b_1, a_2+b_2)$ is defined $\mod(2,2)$ and $\mathrm{ab} = a_1b_1+a_2b_2$. Here in (6)-(8) $x_{\mathrm{a}}\in\tilde{\mathfrak{g}}_{\mathrm{a}}$, $x_{\mathrm{b}}\in\tilde{\mathfrak{g}}_{\mathrm{b}}$, and the element $z\in\tilde{\mathfrak{g}}$ is not necessarily homogeneous. From (6) and (7) it is follows that the general Lie bracket $[\![\cdot,\cdot]\!]$ for homogeneous elements posses two value: commutator $[\cdot,\cdot]$ and anticommutator $\{\cdot,\cdot\}$ as well as in the previous $\mathbb{Z}_2$-case. From (7) it is follows that $\tilde{\mathfrak{g}}_{(0,0)}$ is a Lie subalgebra in $\tilde{\mathfrak{g}}$, and the subspaces $\tilde{\mathfrak{g}}_{(1,1)}$, $\tilde{\mathfrak{g}}_{(1,0)}$ and $\tilde{\mathfrak{g}}_{(0,1)}$ are $\tilde{\mathfrak{g}}_{(0,0)}$-modules:

$$[\tilde{\mathfrak{g}}_{(0,0)},\tilde{\mathfrak{g}}_{(0,0)}] \subseteq \tilde{\mathfrak{g}}_{(0,0)}, \quad [\tilde{\mathfrak{g}}_{(0,0)},\tilde{\mathfrak{g}}_{(a,b)}] \subseteq \tilde{\mathfrak{g}}_{(a,b)}, \tag{10}$$

It should be noted that $\tilde{\mathfrak{g}}_{(0,0)}\oplus\tilde{\mathfrak{g}}_{(1,1)}$ is a Lie subalgebra in $\tilde{\mathfrak{g}}$ and the subspace $\tilde{\mathfrak{g}}_{(1,0)}\oplus\tilde{\mathfrak{g}}_{(0,1)}$ is a $\tilde{\mathfrak{g}}_{(0,0)}\oplus\tilde{\mathfrak{g}}_{(1,1)}$-module, and moreover $\{\tilde{\mathfrak{g}}_{(1,1)}\tilde{\mathfrak{g}}_{(1,0)}\}\subset\tilde{\mathfrak{g}}_{(0,1)}$ and vice versa $\{\tilde{\mathfrak{g}}_{(1,1)},\tilde{\mathfrak{g}}_{(0,1)}\}\subset\tilde{\mathfrak{g}}_{(1,0)}$.

Let us take the complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra (LSA) $\mathfrak{g}$, as a linear space over $\mathbb{C}$:

$$\tilde{\mathfrak{g}} = \bigoplus_{\mathrm{a}=(a_1,a_2)} \tilde{\mathfrak{g}}_{\mathrm{a}} = \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)} \oplus \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)} \qquad (11)$$

and apply to it the following cross-dressing

$$\begin{aligned}\tilde{\mathfrak{g}}' &= \big(e\tilde{\mathfrak{g}}_{(0,0)} \oplus c_1 c_2 \tilde{\mathfrak{g}}_{(1,1)}\big) \oplus \big(c_1 \tilde{\mathfrak{g}}_{(1,0)} \oplus c_2 \tilde{\mathfrak{g}}_{(0,1)}\big) \\ &= \tilde{\mathfrak{g}}'_0 \oplus \tilde{\mathfrak{g}}'_1.\end{aligned} \qquad (12)$$

Because of the Lie symmetries $AdS_5$ and $dS_5$ are real forms of the simple complex Lie algebra $\mathfrak{so}(5;\mathbb{C}) \simeq \mathfrak{sp}(4;\mathbb{C})$ it is natural to belive that $AdS_5$ and $dS_5$ $(N=1)$ supersymmeries are the real forms of the Lie superalgebra $\mathfrak{osp}(1|4;\mathbb{C})$ that is a minimal superextension of $\mathfrak{sp}(4;\mathbb{C})$.

# 3. Bosonic realization of the complex $\mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(1|4;\mathbb{C})$

Let $Bo_2$ be the standard bosonic algebra generated by two oscillators $b_i^+$, $b_i$ $(i = 1, 2)$:

$$[b_i, b_j] = [b_i^+, b_j^+] = 0, \quad [b_i, b_j^+] = \delta_{ij} I. \tag{13}$$

Let's put

$$X_i = \frac{1}{\sqrt{2}} b_i^+, \quad X_{-i} = \frac{1}{\sqrt{2}} b_i, \tag{14}$$

$$X_{ii} = \{X_i, X_i\} = (b_i^+)^2, \quad X_{-i-i} = \{X_{-i}, X_{-i}\} = (b_i)^2, \tag{15}$$

$$X_{i-i} = \{X_i, X_{-i}\} = \frac{1}{2}(b_i^+ b_i + b_i b_i^+), \tag{16}$$

$$X_{1-2} = \{X_1, X_{-2}\} = b_1^+ b_2, \quad X_{2-1} = \{X_2, X_{-1}\} = b_2^+ b_1, \tag{17}$$

$$X_{12} = \{X_1, X_2\} = b_1^+ b_2^+, \quad X_{-2-1} = \{X_{-2}, X_{-1}\} = b_2 b_1. \tag{18}$$

These formulas represent the well-known realization of the $\mathbb{Z}_2$-graded orthosymplectic superalgebra $\mathfrak{osp}(1|4;\mathbb{C})$ in the terms of the $\mathbb{Z}_2$-graded bosonic algebra $Bo_2$.
So we can say that: *A two-oscillator boson system over a complex field $\mathbb{C}$, where bosons in different oscillators commute, generates an orthosymplectic complex contragredient $\mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(1/4;\mathbb{C})$. This superalgebra is a maximally-generated finite-dimensional contragredient $\mathbb{Z}_2$-graded Lie superalgebra for a given two-oscillator boson system, and the all graded Lie superalgebra $Bo_2$ realizes a universally enveloping superalgebra $U(\mathfrak{osp}(1/4;\mathbb{C}))$.*

.
The corresponding cross-dressing bosonic algebra $\tilde{Bo}_2$ is generated by the following elements $\tilde{b}_i^+$, $\tilde{b}_i$ $(i=1,2)$:

$$\{\tilde{b}_1,\tilde{b}_2\}=\{\tilde{b}_1^+,\tilde{b}_2^+\}=0,\quad [\tilde{b}_i,\tilde{b}_i^+]=I\quad (i=1,2). \tag{19}$$

Let's put

$$\tilde{X}_i=\frac{1}{\sqrt{2}}\tilde{b}_i^+,\quad \tilde{X}_{-i}=\frac{1}{\sqrt{2}}\tilde{b}_i, \tag{20}$$

$$\tilde{X}_{ii}=\{\tilde{X}_i,\tilde{X}_i\}=(\tilde{b}_i^+)^2,\quad \tilde{X}_{-i-i}=\{\tilde{X}_{-i},\tilde{X}_{-i}\}=(\tilde{b}_i)^2, \tag{21}$$

$$\tilde{X}_{i-i}=\{\tilde{X}_i,\tilde{X}_{-i}\}=\frac{1}{2}(\tilde{b}_i^+\tilde{b}_i+\tilde{b}_i\tilde{b}_i^+), \tag{22}$$

$$\tilde{X}_{1-2}=[\tilde{X}_1,\tilde{X}_{-2}]=\tilde{b}_1^+\tilde{b}_2,\quad \tilde{X}_{2-1}=[\tilde{X}_2,\tilde{X}_{-1}]=\tilde{b}_2^+\tilde{b}_1, \tag{23}$$

$$\tilde{X}_{12}=[\tilde{X}_1,\tilde{X}_2]=\tilde{b}_1^+\tilde{b}_2^+,\quad \tilde{X}_{-2-1}=[\tilde{X}_{-2},\tilde{X}_{-1}]=\tilde{b}_2\tilde{b}_1. \tag{24}$$

So we can say that: *A two-oscillator boson system over a complex field $\mathbb{C}$, where bosons in different oscillators anticommute, generates a complex contragredient orthosymplectic $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superalgebra $\mathfrak{o\tilde{s}p}(1|2,2;\mathbb{C})$. This superalgebra is a maximally-generated finite-dimensional contragredient $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra for a given two-oscillator boson system, and the total graded Lie superalgebra $\tilde{Bo}_2$ realizes a universally enveloping superalgebra $U(\mathfrak{o\tilde{s}p}(1|2,2;\mathbb{C}))$.*

The orthosymplectic superalgebra $\mathfrak{osp}(1|2n;\mathbb{C})$ can be represented as a direct sum of spaces

$$\mathfrak{osp}(1|2n;\mathbb{C}) = \bigoplus_{1\leq i\leq n} \mathfrak{osp}_i(1|2;\mathbb{C}) \bigoplus_{1\leq i<j\leq n} \mathcal{P}_{ij}, \tag{25}$$

where the linear envelope

$$\mathcal{P}_{ij} = \mathrm{Lin}\{X_{ij}, X_{-i-j}, X_{i-j}, X_{-ij}\} \tag{26}$$

is called the complex space of a curved four-momentum for each fixed $i < j$. In the case of a complete cliffonic cross-dressing for the boson realization of $\mathfrak{osp}(1|2n;\mathbb{C})$ we obtain for the complex $\mathbb{Z}_2 \times ... \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(1|2, ..., 2;\mathbb{C})$.

## 6a. Clifford algebra of quaternionic type

Let us to consider the Clifford algebra of quaternionic type $C_{\bar{2}}$. It generates by the cliffonnic elements $\{e,\, c_1,\, c_2, c_1c_2\}$. We use the standard quatenionic notations:

$$i = c_1, \quad j = c_2, \quad k = c_1c_2 = ij, \tag{27}$$

These generators satisfy the relations:

$$i^2 = j^2 = k^2 = -1. \tag{28}$$

There are two type of quaternion involutions: anti-Hermitian ($^+$) and Hermitian ($^\star$) :

$$i^+_{-H} = -i_{-H}, \quad j^+_{-H} = -j_{-H}, \quad k^+_{-H} = -k_{-H} \tag{29}$$

$$i^\star_H = i_H, \quad j^\star_H = j_H, \quad k^\star_H = k_- \tag{30}$$

## *6b. Real forms of the boson algebra*

$$[b, b^{+}] = I, \tag{31}$$

$$(b)^{+} = b^{+}, \quad (b^{+})^{+} = b \quad (non - compact \quad form), \tag{32}$$

$$(b)^{\star} = i_H b^{+}, \quad (b^{+})^{\star} = -i_H b \quad (compact \quad form) \tag{33}$$

for Hermitian $i_H$:

$$i_H^{\star} = i_H, \quad i_H^2 = -1. \tag{34}$$

## *6c. Real forms of the orthosymplectic supersymmetry* $\mathfrak{osp}(1|2;\mathbb{C})$

Let's put

$$X_{+} = \frac{1}{\sqrt{2}}b^{+}, \quad X_{-} = \frac{1}{\sqrt{2}}b, \tag{35}$$

$$X_{++} = \{X_{+}, X_{+}\} = (b^{+})^{2}, \quad X_{--} = \{X_{-}, X_{-}\} = (b)^{2}, \tag{36}$$

$$X_{+-} = \{X_{+}, X_{-}\} = \frac{1}{2}(b^{+}b + bb^{+}). \tag{37}$$

This is the bosonic realization of the superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$. It is not difficult to verify that the operation $(^{+})$ is involutive on the superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$ and it gives the non-compact real form $\mathfrak{osp}(1|\mathfrak{sp}(2;\mathbb{R}))$:

$$X_{+}^{+} = X_{-}, \quad X_{-}^{+} = X_{+}, \tag{38}$$

$$X_{++}^{+} = X_{--}, \quad X_{--}^{+} = X_{++}, \quad X_{+-}^{+} = X_{+-}. \tag{39}$$

Similarly, in the case of evolution $(^{\star})$ we obtain a compact real form $\mathfrak{osp}(1|\mathfrak{sp}(2))$:

$$X_{+}^{\star} = -i_{H}X_{-}, \quad X_{-}^{\star} = i_{H}X_{+}, \tag{40}$$

$$X_{++}^{\star} = -X_{--}, \quad X_{--}^{\star} = -X_{++}, \quad X_{+-}^{\star} = X_{+-}. \tag{41}$$

## *6d. Real forms of the orthosymplectic supersymmetry* $\mathfrak{osp}(1|4;\mathbb{C})$

$$\mathfrak{osp}(1|4;\mathbb{C}) = \mathfrak{osp}_1(1|2;\mathbb{C}) \oplus \mathcal{P}_{12} \oplus \mathfrak{osp}_2(1|2;\mathbb{C}), \tag{42}$$

where the linear envelope

$$\mathcal{P}_{12} = \mathrm{Lin}\{X_{12}, X_{-2-1}, X_{1-2}, X_{2-1}\} \tag{43}$$

is called the complex space of a curved four-momentum.
The orthosymplectic $\mathbb{Z}_2$-graded supersubalgebras $\mathfrak{osp}_i(1|2;\mathbb{C})$ $(i = 1, 2)$ are generated by the bases (35)-(37).

Basic Lie superalgebras

$$A(m,n),\ \ B(m,n),\ \ C(n+1),\ \ D(m,n), \tag{44}$$

$$F(4),\ \ G(3),\ \ D(2,1;\alpha) \tag{45}$$

Using the standard contraction procedure: $L_{\mu 4} = R\, P_\mu$ $(\mu = 0, 1, 2, 3)$, $Q_k \to \sqrt{R}\, Q_k$ and $\bar{Q}_k \to \sqrt{R}\, \bar{Q}_k$ $(k = 1, 2)$ for $R \to \infty$ we obtain the super-Poincarè algebra (standard and alternative) which is generated by $L_{\mu\nu}$, $P_\mu$, $Q_\alpha$, $Q_{\dot{\alpha}}$ where $\mu, \nu = 0, 1, 2, 3$; $\alpha, \dot{\alpha} = 1, 2$ with the relations (we write down only those which are changed in the standard and alternative Poincarè SUSY.
(I) *for the standard Poincarè SUSY:*

$$[P_\mu, Q_\alpha] \;=\; [P_\mu, \bar{Q}_{\dot{\alpha}}] = 0\;, \quad \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} \;=\; 2\sigma^\mu_{\alpha\beta} \P_\mu, \tag{46}$$

(II) *for the altrenative Poincarè SUSY:*

$$\{P_\mu, Q_\alpha\} \;=\; \{P_\mu, \bar{Q}_{\dot{\alpha}}\} = 0\;, \quad [Q_\alpha, \bar{Q}_{\dot{\alpha}}] \;=\; 2\sigma^\mu_{\alpha\beta} P_\mu. \tag{47}$$